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Note

Partial Vindication of the Bilinear Velocity, Piecewise Constant Pressure Element*

The purpose of this note is explained in the title. This element, hereafter referred to as the 4/1 element, has been the focal point of much criticism (e.g., [1-8]), a good deal of it our own [9-11], when used to solve the incompressible Navier-Stokes equations. Herein we will show, on meshes composed of rectangles, that the 4/1 can be at least as accurate as a MAC-based scheme, contrary to the conclusions drawn by Piva *et al.* [4]. The vehicle for this comparison will be the lid-driven cavity, for which we will compare results at $Re = 1$ (following Piva *et al.*) and at $Re = 400$, following Cullen [5], whose recent publication in this Journal prompted this investigation. In both cases, the tabulated fine mesh results of Winters and Cliffe [12] will be used as the truth—as was done in both of the above references; they appear as the solid curves in all the figures.

We begin with $Re = 1$ and compare three results from a uniform 10×10 element mesh in Fig. 1. The important point to be made here is that the flow is too weak when the "leaky lid" boundary conditions (BCs) are used; i.e., the triangles in Fig. 1

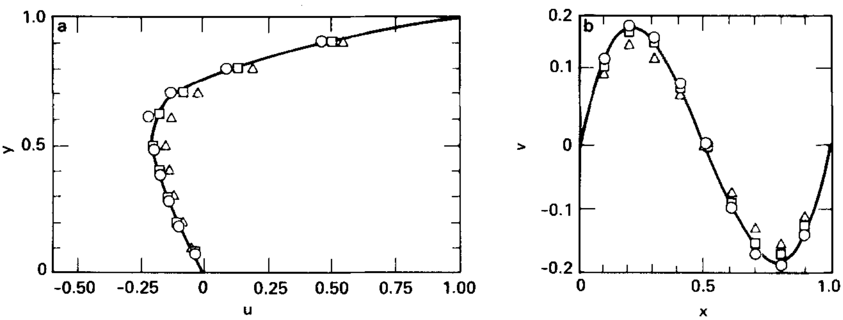

FIG. 1. Velocity profiles through the cavity center for $Re = 1$. △: The 4/1 element with leaky boundary conditions; ○: the 4/1 element with non-leaky conditions; □ the MAC-based scheme of Piva *et al.* (a) u vs y at $x = 0.5$. (b) v vs x at $y = 0.5$.

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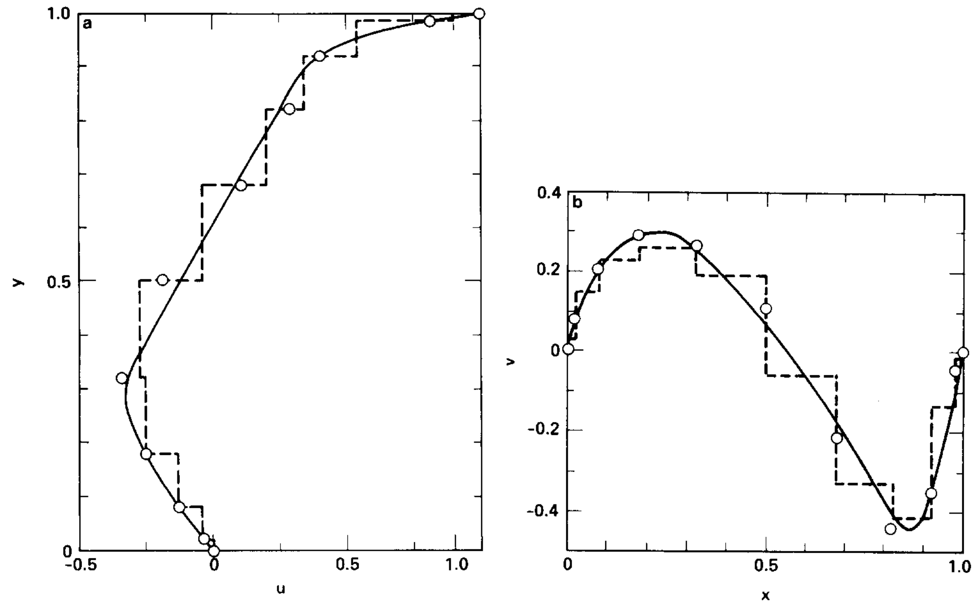

FIG. 2. Same as Fig. 1 except Re = 400. Dashed lines: Cullen's MAC-based results; ○: the 4/1 element results. (a) u vs y at $x = 0.5$. (b) v vs x at $y = 0.5$.

are obtained by enforcing $U = 1$ at all (11) nodes across the cavity top, the same BCs (apparently) used by Piva *et al.* in their finite element simulation. This causes a net mass flux through the box of 0.05—a large leak when it is realised that the total flux (i.e., maximum value of stream function) flowing around the closed cavity is only ~ 0.10. In contrast, the circles show the much better results when non-leaky BCs are employed; viz., $U = 0, 0.5, 1, 1, \ldots, 1, 0.5, 0$. These are seen to be as good, or better, than those from the MAC-based scheme, which also uses non-leaky BCs (Admittedly, it must be known how to employ non-leaky BCs without violating the solvability condition associated with the checkerboard (CB) pressure mode, but this was carefully explained in Sani *et al.* [9].)

We now move on to a more significant Reynolds number (400) and respond to the statement in Cullen [5], "It would be interesting to repeat this calculation with the 4/1 element and the same grid." The results in Figs. 2–5 compare this element (with the non-leaky BC) with one of Cullen's MAC-based schemes on the same graded mesh of 10×10 elements. (We chose the results from his vorticity-based scheme (called V in Cullen [5]), since he seemed to favor it slightly.) For this graded mesh, the consistent non-leaky BCs are given by $u = 0, 0.25, 1, 1, \ldots, 1, 0.25, 0$. The velocity profiles in Fig. 2 show that the 4/1 element is at least as accurate as the MAC-based scheme. The pressure distribution (Fig. 3), and that of the stream function (Fig. 4) at $y = 0.5$, are clearly more accurate than those from the MAC-

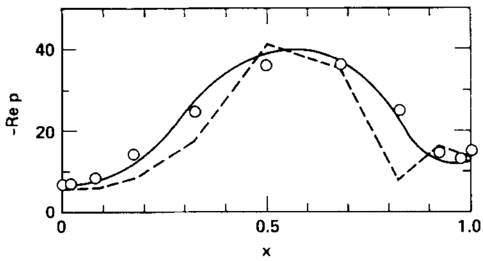

FIG. 3. Pressure distribution at cavity mid-height ($y = 0.5$). Dashed lines: Cullen's MAC-based scheme; ○: the 4/1 element results.

based scheme. The pressures are obtained by properly filtering the spurious CB pressure mode [9]; this is basically area-weighted averaging. The stream function was obtained by simple contour integration of the velocity field along element boundaries. Finally, the vorticity results are compared in Fig. 5; the 4/1 element is, again, at least as accurate as the schemes used by Cullen. This derived variable was obtained from the velocity field via Scheme 2 described by Lee *et al.*, [13]—basically a modified least-squares technique in which the lumped mass matrix is employed to generate a C^0 bilinear fit to the C^{-1} data. It should be pointed out, however, that some of the differences may be related to different post-processing schemes since the resolution is low and the variation normal to the cross-sections is non-trivial.

The main conclusion from this exercise is obvious: While the 4/1 element does have some problems, being less accurate than MAC-based scheme on grids composed of rectangles is not one of them. Another conclusion is that leaky BCs should not be used for driven cavity simulations.

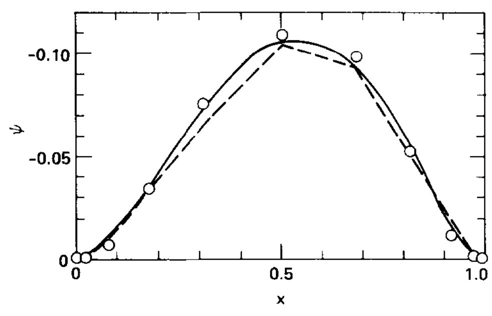

FIG. 4. Same as Fig. 3 except stream function.

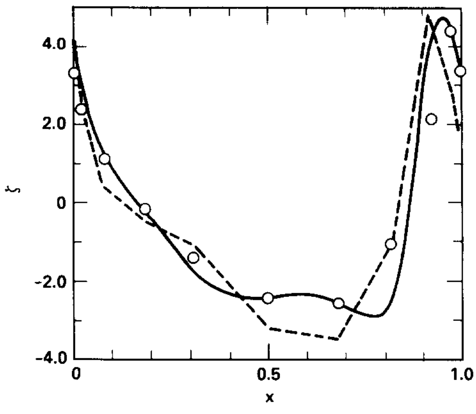

FIG. 5. Same as Fig. 3 except vorticity.

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